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The $d = 6$, $(2, 0)$-tensor multiplet coupled to self-dual strings

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Abstract: We show that the central charges that group theory allows in the $(2, 0)$-supersymmetry translations algebra arise from a string and a 3-brane by commuting two supercharges. We show that the net force between two such parallel strings vanishes. We show that all the coupling constants are fixed numbers, due to supersymmetry, and self-duality of the three-form field strength. We obtain a charge quantization for the self-dual field strength, and show that when compactifying on a two-torus, it reduces to the usual quantization condition of $N = 4$ SYM with gauge group $SU(2)$, and with coupling constant and theta angle given by the τ-parameter of the two-torus, provided that we pick that chiral theory which corresponds to a theta function with zero characteristics, as expected on manifolds of this form.

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1 Introduction

It is believed that $N = 4$ super Yang-Mills theories in four dimensions has its origin in $(2, 0)$ supersymmetric six-dimensional theories [1] [13]. The S-duality property of the $N = 4$ theory would then have a purely geometrical explanation as being the modular group of a two-torus when compactifying the six-dimensional theory to four dimensions. It is however not possible to proceed straightforwardly and reduce the action of the six-dimensional theory since there does not exist any covariant action for a self-dual three-form field strength which is in the $(2, 0)$ tensor multiplet. What one can do is to reduce the equations of motion. By integrating the self-dual field strength (divided by some number) over spatial three-cycles in a non-trivial topology one gets a quantity which is either integer or integer shifted by $1/2$, depending on which chiral theory one has. The various chiral theories can be labeled by the characteristics α and β in $(\frac{1}{2}\mathbf{Z})^{\frac{1}{2}b_3}$, where b_3 is the dimension of the third homology group of the six-manifold or the third Betti number. One can reduce the self-dual gauge potential to four dimensions. Due to its self-duality it reduces to one (compact) scalar and one gauge field. By compactifying on a T^2 with modular parameter τ one should get the charge quantization in four dimensions with a theta-angle θ [3]. The Yang-Mills coupling constant g_{YM} and the theta angle should combine to the τ of the T^2 as $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g_{YM}^2 \hbar}$. We will see that this is true, but only for the the theory with zero characteristics. This is in agreement with the observation that on manifolds with one circle being time and one (or several, in this case two) one-cycle(s) being time-like, the only theory which can candidate to give a modular invariant partition function is that with zero characteristics [9], [10]. One should perhaps not expect full modular invariance of the partition function only for the tensor part, but one should expect that this partition function transform to itself at least up to a phase factor and that is the case only for the theory with zero characteristics.

The free $(2, 0)$-theory has no adjustable parameters. Their numerical values are determined from the $(2, 0)$-supersymmetry up to an overall coupling constant. This overall coupling constant, which we will call λ, can only take one particular value, but that does not follow from supersymmetry. We have found two seemingly unrelated ways to determine its value, or more precisely, the ratio λ/g where g is the unit in which the self-dual charges are quantized. The first criterion is that there should only be finitely many chiral theories. The second criterion is that the Wilson surface observables, $\exp 2\pi i \int_D \frac{H^+}{g}$, over three-dimensional surfaces D, should commute, in order for the $U(1)$ Wilson and 't Hooft lines which one obtains when reducing to four dimensions, to commute. We do not know how to write an observable in six dimensions that reduces to $SU(N)$ Wilson and 't Hooft lines in four dimensions.

In section 2 we examine how $(2, 0)$ supersymmetry constrains the parameters in an action. We write an action for a non-self-dual gauge field, from which the equation of motion for the self-dual part, H^+, of the field strength can be obtained by decomposing H as $H = H^+ + H^-$. Supersymmetry fixes the sizes of the parameters in this action only up to an overall factor, which, as we will see in section 3 and 4, is determined from the self-duality of the field strength. We construct supercharges out of the fields in the $(2, 0)$ tensor multiplet. When we anti-commute two supercharges, in the same manner as in [2], we find central charges which correspond to a string and a 3-brane, respectively. We use the BPS-condition on the string tension to fix the relative size of the constants in the action which describes a tensor multiplet that couples to strings. We show that the net force between two equally charged parallel strings vanishes due to attraction via scalars and repulsion due to the self-dual tensor field.

In section 3 we examine how self-duality of the field strength constrains the value that the coupling constant takes, given a time direction. The natural framework for this is the Hamiltonian formulation. The condition we want to satisfy is that the partition function for the non-chiral two-form potential shall be possible to holomorphically factorize into $\frac{1}{2}b_3$ number of terms.

In section 4 we show that the value we have obtained of the coupling is precisely that which gives the 'correct' commutation relations of the Wilson surface observables.

In section 5 we obtain the usual quantization conditions with a theta-angle of $N = 4$ SYM with gauge group $SU(2)$ spontaneously broken to $U(1)$ by compactifying a $(2,0)$-theory with one massless tensor multiplet and with zero characteristics, on $T^2 \times M_4$, where $M_4 = S^1 \times M_3$. The S^1 is time.

2 Coupling of the tensor multiplet to a classical string

In this section we will assume that we have a flat six-dimensional background with metric $G_{\mu\nu} = \mathrm{diag}(-1,1,1,1,1,1)$. We will use $\mu = \{0, i\}... = 0, 1, ...5$ as vector indices and $A, B, ...$ as Dirac spinor indices of the Dirac representation $8 = 4 \oplus 4'$, and $\alpha, \beta, ...$ and $\alpha', \beta', ...$ as the Weyl spinor indices respectively, in the Lorentz group $SO(1,5)$; $a, b, ... = 1, 2, ..., 5$ as vector indices and $i, j, ...$ as spinor indices in the R-symmetry group $SO(5)_R$. More conventions about our spinors are found in the appendix A. We define the three-form field strength H from the two-form gauge potential B as $H = dB$. Here $H = \frac{1}{3!}H_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho$ and $dB = \frac{1}{2!}\partial_\mu B_{\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho$. The components of the field strength are thus

$$H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} \tag{1}$$

We note that there does not exist a decomposition of the gauge potential B into chiral potentials $B^\pm$ unless the fields satisfy the equation of motion. If $H = dB^+ + dB^-$ where $*dB^\pm = \pm dB^\pm$ then $d*H = dH = 0$. Conversely, if $dH = d*H = 0$, then we can locally write $H = dB$ and $*H = d\tilde{B}$ and hence, locally, we have that $H = dB^+ + dB^-$ where we can take $B^\pm = \frac{1}{2}(B \pm \tilde{B})$. We will in this paper always assume that the fields are on-shell so that such chiral potentials exist (locally).

The supersymmetry charges of the $d = 6$, $(2,0)$-theory transform in the representation $(4,4)$ of $SO(1,5) \times SO(5)$. The anti-commutator of two such supercharges will transform in the representation (s (a) means the (anti) symmetric part)

$$((4,4)\times(4,4))_s \simeq (6_a \oplus 10^+_s, 1_a \oplus 5_a \oplus 10_s)_s = (6_a, 1_a) \oplus (6_a, 5_a) \oplus (10^+_s, 10_s) \simeq P_\mu \oplus Z_{\mu a} \oplus W^+_{\mu\nu\rho,ab}, \tag{2}$$

so the most general $SO(1,5) \times SO(5)_R$-invariant supertranslations algebra is [4]

$$\{Q_{\alpha i}, Q_{\beta j}\} = i\left(\Omega_{ij}(\gamma^\mu)_{\alpha\beta}P_\mu + (\sigma^a)_{ij}(\gamma^\mu)_{\alpha\beta}Z_{\mu a} + \frac{1}{2!3!}(\sigma^{ab})_{ij}(\gamma^{\mu\nu\rho})_{\alpha\beta}W^+_{\mu\nu\rho,ab}\right). \tag{3}$$

The overall factor i in the right hand side comes from the symplectic Majorana condition $(Q_{\beta j})^\dagger = i\Omega^{ji}Q_{\alpha i}(\gamma^0)_{\alpha\beta}$. We define the translation generator as $[P_\mu, \cdot] = i\partial_\mu$. From this algebra one derives that there is a massless tensor multiplet on which these supercharges act as [6]

$$\begin{aligned}
[Q_{\alpha i}, B_{\mu\nu}] &= i(\gamma_{\mu\nu})_\alpha{}^\beta\psi_{\beta i} \\
[Q_{\alpha i}, \phi_a] &= i(\sigma_a)_i{}^j\psi_{\alpha j} \\
\{Q_{\alpha i}, \psi_{\beta j}\} &= \frac{i}{24}\Omega_{ij}(\gamma^{\mu\nu\rho})_{\alpha\beta}H^+_{\mu\nu\rho} + \frac{i}{2}(\sigma^a)_{ij}(\gamma^\mu)_{\alpha\beta}\partial_\mu\phi_a.
\end{aligned} \tag{4}$$

The commutator of two variations close only if one uses the equations of motion. The action for this massless multiplet can be determined by requiring that the supercharges transform the massless fields in the tensor multiplet as above. We find that the supercharges [7]

$$Q_{\alpha i} = \frac{1}{6}\int d^5x\gamma^{\mu\nu\rho}\gamma^0\psi_{\alpha i}H^+_{\mu\nu\rho} + \int d^5x\sigma_a\gamma_\mu\gamma^0\psi_{\alpha i}\partial^\mu\phi^a \tag{5}$$

will do the job if and only if the canonical equal-time commutation relations are

$$\begin{aligned} [\phi^a(x), \partial^0\phi_b(y)] &= -i\frac{1}{2}\delta^a_b\delta^5(x-y) \\ \{\psi_{\alpha i}(x), \psi_{\beta j}(y)\} &= i\frac{1}{4}\Omega_{ij}(\gamma_0)_{\alpha\beta}\delta^5(x-y) \\ [H^+_{lmn}(x), H^+_{ijk}(y)] &= i\frac{3}{2}\epsilon_{[ijklm}\partial_{n]}\delta^5(x-y) \end{aligned} \tag{6}$$

which in turn can be derived from the non-chiral action

$$S = \int d^6x(-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \partial_\mu\phi^a\partial^\mu\phi_a + 4\psi^{\alpha i}\Omega_{ij}(\gamma^\mu)_{\alpha\beta}\partial_\mu\psi^{\beta j}). \tag{7}$$

(The last of these commutation relations is a bit tricky and is derived in appendix B.) We will now anti-commute two such supercharges and pay attention only to terms that survive only on topologically non-trivial six-manifolds [2], which will turn out to correspond to the non-compact topologies one gets by deleting an infinite string and a 3-brane respectively from the M5-brane world-volume which we have assumed to be flat, i.e. with vanishing intrinsic curvature [5]. (The extrinsic curvature, that is, how the M5-brane is embedded in eleven dimensions, is an other thing which we don't consider here.) We notice that $\gamma^{\mu\nu\rho}H^+_{\mu\nu\rho} = 2\gamma^{ijk}H^+_{ijk}$ due to self-duality. Then we get

$$\begin{aligned} \{Q_{\alpha i}, Q_{\beta j}\} &= \ldots + \frac{i}{6}(\sigma^a)_{ij}(\gamma^{ijkl0})_{\alpha\beta}\int d^5xH_{ijk}\partial_l\phi_a \\ &\quad + \frac{i}{4}(\sigma^{ab})_{ij}(\gamma^{\mu\nu 0})_{\alpha\beta}\int d^5x\partial_\mu\phi_a\partial_\nu\phi_b \\ &= \ldots + i(\sigma^a)_{ij}(\gamma_m)_{\alpha\beta}\int H\wedge d\phi_a\wedge dx^m \\ &\quad + \frac{i}{2!3!}(\sigma^{ab})_{ij}(\gamma_{klm})_{\alpha\beta}\int d\phi_a\wedge d\phi_b\wedge dx^k\wedge dx^l\wedge dx^m \end{aligned} \tag{8}$$

Now we assume that we have an infinite string Σ in the X^5-direction, located at $X^1 = X^2 = X^3 = X^4 = 0$, so that we integrate over the manifold $\mathbf{R}^5 - \mathbf{R} = \mathbf{R}\times(\mathbf{R}^4 - \{0\}) = \mathbf{R}\times(\mathbf{R}_+\times S^3)$. Then we have

$$\begin{aligned} &= \ldots + i(\sigma^a)_{ij}(\gamma_5)_{\alpha\beta}\int_{string} dx^5\int_{S^3\times\mathbf{R}_+} H^+\wedge d\phi_a \\ &= \ldots + i(\sigma^a)_{ij}(\gamma_5)_{\alpha\beta}\int_{string} dx^5 g^+\phi_a\mid_\Sigma \end{aligned} \tag{9}$$

where in the last step we have defined $g^+ \equiv \int_{S^3} H^+$. From this we read off

$$Z_{\mu a} = \int_{string} dx^5 g^+\delta^5_\mu\phi_a\mid_\Sigma . \tag{10}$$

The string tension T is given by

$$\int_{string} T = \sqrt{Z_{\mu a}Z^{\mu a}} \tag{11}$$

for BPS-saturated strings, so

$$T = g^+\sqrt{\phi^a\phi_a}\mid_\Sigma . \tag{12}$$

This tension will contain an infinite part coming from the ϕ-field the string produces itself, plus a finite part coming from ϕ-fields produced by other strings.

Similarly for an infinite 3-brane in the $X^{1,2,3}$ directions, localized at $X^4 = X^5 = 0$, we get the manifold $\mathbf{R^5} - \mathbf{R^3} = \mathbf{R^3} \times (\mathbf{R}_+ \times S^1)$, and

$$= ... + i\frac{1}{2!}(\sigma^{ab})_{ij}(\gamma_{123})_{\alpha\beta} \int_{3-brane} dx^1 \wedge dx^2 \wedge dx^3 \int_{S^1 \times \mathbf{R}_+} d\phi_a \wedge d\phi_b \tag{13}$$

from which we read off

$$W^{\mu\nu\rho}{}_{ab} = \delta^{\mu\nu\rho}_{123} \int_{3-brane} dx^1 \wedge dx^2 \wedge dx^3 \int_{S^1 \times \mathbf{R}_+} d\phi_a \wedge d\phi_b \tag{14}$$

and the three-brane tension,

$$\tau_3 = \int_{S^1 \times \mathbf{R}_+} d\phi_a \wedge d\phi_b. \tag{15}$$

The equations of motion for the bosonic fields in the tensor multiplet coupled to a string can be obtained by adding to the action the interaction terms

$$\sum_i \int_{\Sigma_i} T\sqrt{-h} d^2\sigma, \tag{16}$$

where i runs over all the string world-sheets, and $h_{\alpha\beta}$ is the induced metric on the string world-sheet. For an infinite self-dual BPS-string Σ in the X^5-direction we get

$$-\int_\Sigma T\sqrt{-h} d^2\sigma = -g^+ \int d^6x |\phi(x)| \int_\Sigma d^2\sigma \delta^6(x - X(\sigma)) \tag{17}$$

where $|\phi| \equiv \sqrt{\phi^a \phi_a}$ and we get the equation of motion

$$2\partial_i \partial^i |\phi(x)| = g^+ \int_\Sigma d^2\sigma \delta^6(x - X(\sigma)). \tag{18}$$

The equations of motion for the self-dual field strength are

$$\begin{aligned} *d * H^+ &= J \\ *dH^+ &= J \end{aligned} \tag{19}$$

where the current J^+ is given by

$$J^{\mu\nu} = \sum_i g^+ \int_{\Sigma_i} dX^\mu \wedge dX^\nu = \sum_i g^+ \int_{\Sigma_i} \sqrt{-h}\frac{1}{2!}\varepsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \delta^6(x - X(\sigma)). \tag{20}$$

We now see that g^+ is the electric and the magnetic charge of this string, which means that we have a self-dual string. The B-field from an infinite string at $X^{1,2,3,4} = 0$ thus satisfies the equation of motion

$$\partial_i \partial^i B^+_{\mu\nu}(x) = g^+ \delta^{05}_{\mu\nu} \int_\Sigma d^2\sigma \delta^6(x - X(\sigma)). \tag{21}$$

Both of the equations of motion (18) and (21) reduce to a four-dimensional equation of the form

$$\partial_i \partial^i f(\mathbf{x}) = \delta^4(\mathbf{x}) \tag{22}$$

which has the solution

$$f(\mathbf{x}) = f(\infty) - \frac{1}{2\pi^2 |\mathbf{x}|^2}. \tag{23}$$

The equation of motion for a second string with the same charges as the first one, which moves in given fields of the tension $T(x)$ and the potential $B^+_{\mu\nu}(x)$, is most easily derived by varying the Polyakov string action coupled to B^+,

$$-\frac{1}{2}\int d^2\sigma T(X)\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu + g^+\int d^2\sigma\frac{1}{2!}\sqrt{-\gamma}\varepsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X) \tag{24}$$

with respect to X^μ, and then putting the auxiliary metric $\gamma_{\alpha\beta}$ equal to the induced metric $h_{\alpha\beta}$. We then get

$$\begin{aligned} T\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta X_\rho) &= \frac{1}{2}\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu\partial_\rho T - \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\rho\partial_\mu T \\ &\quad -\frac{g^+}{4}\epsilon^{\alpha\beta}(\partial_\mu B^+_{\nu\rho} + \partial_\nu B^+_{\rho\mu} + \partial_\rho B^+_{\mu\nu})\partial_\alpha X^\mu\partial_\beta X^\nu \end{aligned} \tag{25}$$

which, in the case of a straight string parallel with the first one, reduces to

$$T\partial^\alpha\partial_\alpha X_\rho = \partial_\rho T - \frac{g^+}{2}\partial_\rho B^+_{05} = \frac{{g^+}^2}{2\pi}\frac{x_\rho}{|\mathbf{x}|^4} - \frac{{g^+}^2}{2\pi}\frac{x_\rho}{|\mathbf{x}|^4} = 0. \tag{26}$$

That is, the attractive force due to interaction via scalars ϕ_a cancels the repulsive force via gauge bosons $B^+_{\mu\nu}$ if the two strings are parallel. If the strings instead had been anti-parallel the forces would have added up, since the Lorentz force would change sign. If one compactify the x^5 direction then orientation reversal of the string becomes tantamount to changing the sign of its charge.

3 Holomorphic factorization of the partition function

In this section we will consider compact topologically non-trivial six-manifolds of the form $M_6 = S^1 \times M_5$ where M_5 is some compact five-manifold. Having assumed this, it is possible to define a basis of the homology group $H_3(M_6, \mathbf{Z})$ consisting A-cycles $\{a_i\}$ that wind around the circle and B-cycles $\{b_j\}$ that do not wind around the circle. They are dual in the sense that they have intersection numbers $a_i \cdot a_j = b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_i^j$. We will let the circle be in the time-direction, so in particular the B-cycles will be spatial. We define a basis $[E_A^i]$ and $[E_B^i]$ of $H^3(M_6, \mathbf{Z})$, which is dual to $H_3(M_6, \mathbf{Z})$, that is, $\int_{a_i} {E_A}^j = \int_{b_i} {E_B}^j = \delta_i^j$ and $\int_{b_i} {E_A}^j = \int_{a_i} {E_B}^j = 0$. It will be symplectic, $\int_{M_6} {E_B}^i \wedge {E_A}^j = \delta_i^j$. We will take E_A and E_B to be harmonic representatives.

We want to make use of a complex structure given by the Hodge duality operator, *, on the intermediate Jacobian $H^3(M_6, \mathbf{R})/H^3(M_6, \mathbf{Z})$. [8] But this is possible only if $*^2 = -1$. This forces us to make a Wick rotation, $x^0 \to x^0_E = ix^0$, such that M_6 becomes an Euclidean manifold. We define the period matrix $Z = X + iY$ with the matrices X and Y having real entries, by declaring

$$\begin{aligned} E^+ &= ZE_A + E_B \\ E^- &= \bar{Z}E_A + E_B. \end{aligned} \tag{27}$$

to be self-dual and anti-self-dual respectively. More explicitly this means that

$$E_B = -XE_A + Y * E_A. \tag{28}$$

For convenience we have defined our basis forms such that $\int_{b_i} {E_\pm}^j = \delta_i^j$.

We can also introduce a symplectic basis for the space of those exact three-forms whose (anti-)self-dual parts also are exact. We denote these basis elements as E'_A and E'_B, and they fulfil $\int_{M_6} E'_B \wedge E'_A = 1$. We can define a matrix Z' such that $E'^+ = Z'E'_A + E'_B$ is self-dual. Then $E'^- = \bar{Z}'E'_A + E'_B$ will be anti-self-dual. Together with the harmonic three-forms these exact forms span the whole space of solutions of the equation of motion $dH = d*H = 0$, where H is a three-form field strength. We can divide the field strength into a zero-mode part and an oscillator part, $H = H_0 + H_{osc}$, and write mode-expansions in the bases of zero-modes E_A and E_B and oscillator modes E'_A and E'_B respectively, as

$$\begin{aligned} H_0 &= h^t_A E_A + h^t_B E_B \\ H_{osc} &= {h'}^t_A {E'}_A + {h'}^t_B {E'}_B. \end{aligned} \tag{29}$$

The (zero-)mode expansions we will be most interested in are

$$\begin{aligned} H_0^+ &= {h^+}^t (ZE_A + E_B) \\ H_0^- &= {h^-}^t (\bar{Z}E_A + E_B). \end{aligned} \tag{30}$$

The Lagrangian density for a free non-chiral two-form potential $B_{\mu\nu}$ with Euclidean field strength $H = dB$ is given by

$$\mathcal{L} = -\frac{1}{2\lambda^2} H \wedge *H \tag{31}$$

where λ is a dimensionless coupling constant. When decomposing the field strength as $H = H^+ + H^-$ where $*H^\pm = \pm i H^\pm$, we get

$$\mathcal{L} = -\frac{1}{\lambda^2} i H^- \wedge H^+ = -\frac{iG}{12\lambda^2}\epsilon^{0ijklm}(H^-_{0ij}H^+_{klm} - H^-_{klm}H^+_{0ij}) \tag{32}$$

The momentum conjugate to B_{ij} is then (if we temporarily treat B_{ij} and B_{ji} as independent variables, as in the appendix)

$$\Pi^{ij} = i\frac{1}{2\lambda^2}\sqrt{G}H^{0ij} = i\frac{i}{12\lambda^2}G\epsilon^{0ijklm}(H^+_{klm} - H^-_{klm}) \tag{33}$$

If we make the gauge choice $B_{0i} = 0$, then the Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= i\Pi^{ij}H_{0ij} - \mathcal{L} = -\frac{i}{12\lambda^2}G\epsilon^{0ijklm}(H^+_{klm}H^+_{0ij} - H^-_{klm}H^-_{0ij}) \\ &= -i\frac{1}{\lambda^2}((H^+)_B \wedge (H^+)_A - (H^-)_B \wedge (H^-)_A) \end{aligned} \tag{34}$$

When we quantize we substitute the Poisson-bracket with a commutator. For the zero-mode oscillators we then get the commutation relations

$$[{h^+}^i, {h^+}^j] = 0 \tag{35}$$

as we will see in section 4. A complete treatment would require the commutators of the oscillator opertors, $[{h'^+}^i, {h'^+}^j]$, as well. But we will not need these commutation relations for our purposes. We will only be interested in the zero-mode part of the Hilbert space, which is spanned by eigenvectors $|h^+ >$ of h^+. We divide the Hamiltonian density into a zero-mode part $\mathcal{H}_0$ and a oscillator part $\mathcal{H}_{osc}$. The zero-mode part is given by the operator

$$\mathcal{H}_0 = -\frac{i}{\lambda^2}({h^+}^t E_B \wedge E^t_A Z h^+ - {h^-}^t E_B \wedge E^t_A \bar{Z} h^-). \tag{36}$$

By using the symplectic property of the three-form basis we get

$$\int_{S^1 \times M_5} \mathcal{H}_0 = -\frac{i}{\lambda^2}({h^+}^t Z h^+ - {h^-}^t \bar{Z} h^-). \tag{37}$$

We notice that with Re $Z = 0$ this quantity always is positive, i.e. the energy is positive, as a consequence of the fact that the period matrix always has the property that Im $Z > 0$.

We can extract the zero-mode part by integrating over a spatial cycle b^i,

$$\int_{b_i} \frac{H^\pm}{g} = \frac{h^{\pm i}}{g} \equiv w^{\pm i}. \tag{38}$$

We will define g such that the eigenvalues of

$$w \equiv w^+ + w^- \tag{39}$$

are integers. The minimal magnetic charge is thus assumed to be g. The numerical value of this charge can be determined from the quantization condition [15] for dyonic strings in six dimensions with electric and magnetic charges (e^i, g^i),

$$e^i g^j + e^j g^i = 2\pi\hbar n^{ij}, \tag{40}$$

where $n^{ij} \in \mathbf{Z}$. This is a much stronger condition than the corresponding Dirac-Schwinger-Zwanziger condition in four dimensions, due to the plus sign. In four dimensions there is a minus sign instead, and hence one can draw no conclusions by considering two equally charged dyons in four dimensions. This situation is different in six dimensions. In particular we can have no theta angle in six dimensions. Another restriction comes from the fact that any consistent chiral theory must contain only self-dual strings. By taking two equally charged self-dual strings with charges $(e^i, g^i) = (g, g)$, the charge quantization condition implies that the smallest such charge is given by

$$g^2 = \pi\hbar. \tag{41}$$

At this stage it is not clear that $\int_{b_i} H$, where H is *non*-self-dual, should be quantized in units of g. But we will give an argument for this at the end of this section.

We will now make use of the gauge equivalence of the non-chiral potential $B \simeq B + \Delta B$, where ΔB has periods which are integer multiples of g. This fact is derived in appendix C by using the fact that B is a connection on a gerbe.[1] The operator which implements such a gauge transformation is given by

$$\exp \frac{i}{\hbar} \int_{M^5} d^5x \Pi^{ij} \Delta B_{ij}. \tag{43}$$

This is proved in the appendix B. Now gauge equivalence means that this operator should have eigenvalues one. This implies that, if we choose ΔB such that it has exactly one non-zero period being g over a two-cycle that has as its Poincare dual the three-cycle b_i, then

$$\frac{i}{\hbar} \int_{M^5} d^5x \Pi^{ij} \Delta B_{ij} = -\frac{i}{\hbar\lambda^2} \int_{M^5} (H^+ - H^-) \wedge \Delta B = -\frac{ig^2}{\hbar\lambda^2} (w^+ - w^-)^i \tag{44}$$

is an integer multiple n of $2\pi i$.

[1] A shorter argument can be made if the three-cycle is S^3. Then we need only two covers U_N and U_S over each of which the gauge potentials are uniquely defined, and the complications discussed on triple overlaps in the appendix do not enter. Let $V_{N(S)}$ be adjacent neighbourhoods such that $S^3 = V_N \cup V_S$. From Stokes theorem we then get

$$\int_{S^3} H = \int_{V_N} dB_N + \int_{V_S} dB_S = \int_{\partial V_N} (B_N - B_S) = \int_{\partial V_N} \Delta B \tag{42}$$

which indicates that $\int \Delta B$ over two cycles is quantized in the same units as $\int H$ is over three-cycles.

Now if we choose our coupling constant such that

$$\frac{g^2}{\lambda^2} = \pi\hbar, \tag{45}$$

which, since $g = \sqrt{\pi\hbar}$, means that

$$\lambda = 1, \tag{46}$$

then the zero-mode contribution of the time-integrated Hamiltonian is

$$\int_{S^1 \times M^5} \mathcal{H} = -i\pi({w^+}^t Z w^+ - {w^-}^t \bar{Z} w^-) \tag{47}$$

where $w^\pm = \mp n + \frac{w}{2}$ which is necessary in order to holomorphically factorize the partition function into a finite sum of chiral times anti-chiral partition functions [8]. Each of these chiral partition functions then describe different chiral theories. This should allow us to interpret any of these theta-functions as a trace Tr $\exp{-TH^+}$ where H^+ is (the zero-mode contribution of) the chiral part of the Hamiltonian and T is an Euclidean time interval. We will take this as a part of the definition of H^+.

We have a gauge invariance in the non-chiral theory, which means that we can insert an operator which performs such a gauge transformation without changing the non-chiral partition function. But such an operator will permute the chiral partition functions. We thus have to consider the effect of inserting the operator

$$\exp \frac{i}{\hbar} \int d^5x 2{\Pi^+}^{ij} \Delta {B^+}_{ij}. \tag{48}$$

which transforms the state $|B^+ >$ to $|B^+ + \Delta B^+ >$ as is showed in appendix B. The zero-mode contribution to the chiral partition function is then

$$\begin{aligned} &\sum_{w^+} < w^+ | e^{\frac{i}{\hbar}\int d^5x 2{\Pi^+}^{ij}\Delta {B^+}_{ij}} e^{-\int_{M_6} \mathcal{H}^+} | w^+ > \\ &= \sum_{w^+} < w^+ | e^{\frac{i}{\hbar} 2 \int H^+ \wedge \Delta B^+} e^{-\int_{M_6} \mathcal{H}^+} | w^+ > \\ &= \sum_{w^+} e^{i 2\pi {w^+}^t \beta} e^{i\pi w^+ Z w^+} \end{aligned} \tag{49}$$

where we have defined

$$\beta^i = \int_{\tilde{b}_i} \frac{\Delta B^+}{\sqrt{\pi\hbar}} \tag{50}$$

where $\tilde{b}_i$ is Poincare dual to b_i. We do not know any direct way to deduce over which values w^+ should run in the sum (more than that it should be integer and/or half-integer valued since it is given by $w^+ = n + \frac{w}{2}$). But we know [8] that the answer must be a theta-function $\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (Z)$, and hence we deduce that ${w^+}^i = \int_{b_i} \frac{H^+}{\sqrt{\pi\hbar}} \in \mathbf{Z} + \alpha^i$. Thus the 'physical' field strength (by 'physical' we will mean a field strength which when integrated over a three-cycle gives a magnetic charge) is not quite a connection on a gerbe. We then have to rescale the gauge field as

$$B_{phys} = \frac{1}{2}\sqrt{\frac{\hbar}{\pi}} B_{math} \tag{51}$$

to obtain the quantization condition of a self-dual connection,

$$\int_{b_i} \frac{H^+_{math}}{2\pi} \in \mathbf{Z} + \alpha^i. \tag{52}$$

This is thus the quantization condition one has in in a chiral theory which is characterized by the $\frac{b_3}{2}$-dimensional vectors α and β, with entries in $\frac{1}{2}\mathbf{Z}$. We now see that for $\alpha = 0$, $\int_{b_i} H^+$ is indeed quantized in units of the smallest charge of a self-dual string, $g = \sqrt{\pi\hbar}$ as it should be on a topologically trivial manifold containing one infinite straight self-dual string. We cannot get non-zero α in that case. That requires a more complicated topology than what one obtains by deleting an infinite straight string from an otherwise topologically trivial manifold. Now there could be a problem that the manifold one obtains by deleting strings necessarily is non-compact. In our formalism we assumed the manifold to be compact.

4 Commutation relations of surface observables

In a curved space with Minkowski signature we have the commutation relations (which are derived in appendix B) [11],

$$[H^+_{ijk}(x), H^+_{i'j'k'}(x')] = i\hbar\frac{3}{2}\epsilon_{i'j'k'[ij}\partial_{k]}\delta^5(x-x') \tag{53}$$

for the 'physical' fields. From this we can compute the commutation relation between Wilson surfaces $\int_D \frac{H^+}{\sqrt{\pi\hbar}}$ where D is a three-dimensional surface with boundary $\partial D = \Sigma$,

$$\begin{aligned}
& \left[\int_D \frac{H^+}{\sqrt{\pi\hbar}}, \int_{D'} \frac{H^+}{\sqrt{\pi\hbar}}\right] \\
= & \left[\frac{1}{\sqrt{\pi\hbar}}\int_D \frac{1}{3!}H^+_{ijk}(x)dx^i\wedge dx^j\wedge dx^k, \frac{1}{\sqrt{\pi\hbar}}\int_{D'}\frac{1}{3!}H^+_{i'j'k'}(x')dx'^i\wedge dx'^j\wedge dx'^k\right] \\
= & -\frac{i}{2\pi}\int_{D'}\frac{1}{3!}dx'^i\wedge dx'^j\wedge dx'^k\int_{\partial D}\frac{1}{2!}dx^i\wedge dx^j\epsilon_{i'j'k'ij}\delta^5(x-x') \\
= & -\frac{i}{2\pi}D'\cdot\partial D = -\frac{i}{2\pi}L(\Sigma,\Sigma').
\end{aligned} \tag{54}$$

The dot, $\cdot$, denotes the intersection number and $L(\Sigma, \Sigma')$ is defined as in the last line and is the linking number of the two two-cycles Σ and Σ'.

We now see that the quantities $w^{+i} = \int_{b^i} \frac{H^+}{\sqrt{\pi\hbar}}$ commute if b^i are three-cycles, $\partial b^i = \emptyset$, which justifies our treatment of these quantities as c-number valued 'charges'.

We now consider open curves D with boundary Σ. Associated with such surfaces we define the Wilson surface observables $W(\Sigma) \equiv \exp 2\pi i \int_D \frac{H^+}{\sqrt{\pi\hbar}}$. By using the BCH-formula we see that these observables commute at equal time. We could also have gone backwards and showed that the coupling constant would have to take the value $\lambda = 1$ in order for these surface observables to commute. They should really commute in order to yield correct commution relations when reducing on a two-torus. We then get $U(1)$ gauge theory, and these surface observables become Wilson lines and 't Hooft lines depending on whether the surface wraps the a- or b-cycle of the two-torus. Then the above commutation relation reduces to the old fact that the Wilson and 't Hooft lines commute in $U(1)$-gauge theory [12]. We think it is remarkable that these two entirely different ways of computing the coupling constant yield the same answer. Using Wilson lines to compute λ did not require a non-trivial topology as holomorphic factorization did.

5 Reduction to four dimensions

We will now start from a Minkowski six-manifold and make dimensional reduction by letting $x^{4,5} \in [0,1]$ be coordinates on a two-torus and x^i $(i = 0,1,2,3)$ be the remaining coordinates [13]. We will denote the moduli parameter of the torus as $\tau = \tau_1 + i\tau_2$. In this section we will use the mathematicians conventions for the gauge fields so that they will be connections on a 1-gerbe and 0-gerbe (line-bundle) respectively. This convention has the advantage that it makes the S-duality transformations look nicer. We dimensionally reduce the on-shell self-dual field strength as

$$H^+ = \frac{1}{3!}H^+_{ijk}dx^i \wedge dx^j \wedge dx^k + \frac{1}{2!}F_{ij}dx^i \wedge dx^j \wedge dx^4 + \frac{1}{2!}\tilde{F}_{ij}dx^i \wedge dx^j \wedge dx^5 + \partial_i B^+_{45}dx^i \wedge dx^4 \wedge dx^5. \quad (55)$$

Due to self-duality, H^+_{ijk} and $\partial_i B^+_{45}$ are related. Likewise $F = dA$ and $\tilde{F}$ are related as

$$\tilde{F} = -\tau_1 F + \tau_2 * F \quad (56)$$

if we define τ as

$$dx^4 = \tau_1 dx^5 - \tau_2 * dx^5. \quad (57)$$

Invariance under diffeomorphisms implies in particular invariance under modular transformations of the T^2. For H^+ to be invariant we must then impose the following transformation rules, $x^4 \to x^5$, $x^5 \to -x^4$, $\tau \to -\frac{1}{\tau}$, $F \to \tilde{F}$, $\tilde{F} \to -F$ and $x^4 \to x^4 + x^5$, $\tau \to \tau + 1$, $\tilde{F} \to \tilde{F} - F$. This diffeomorphism invariance is S-duality from the four-dimensional point of view.

We will now integrate H^+ over three-cycles $\Sigma \times \gamma$ where Σ is a two-cycle not on the torus, and γ is either the a or the b-cycle of the torus, normalized such that $\int_a dx^4 = \int_b dx^5 = 1$, $\int_a dx^5 = \int_b dx^4 = 0$. We then get

$$\begin{pmatrix} \int_{\Sigma \times a} H^+ \\ \int_{\Sigma \times b} H^+ \end{pmatrix} = \begin{pmatrix} \int_\Sigma F \\ \int_\Sigma \tilde{F} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\tau_1 & \tau_2 \end{pmatrix} \begin{pmatrix} \int_\Sigma F \\ \int_\Sigma *F \end{pmatrix}. \quad (58)$$

In section 3 we saw that $\int_{\Sigma\times\gamma} \frac{H^+}{2\pi} = w_\gamma$ where w^+_γ is either in $\mathbf{Z}$ or in $\mathbf{Z}+\frac{1}{2}$ depending on which theory we are looking at (i.e. on which theta function we pick). Now we should choose the theory which corresponds to the theta function with zero characteristics, $\theta\left[\begin{smallmatrix}00\cdots0\\00\cdots0\end{smallmatrix}\right]$, which was found in [14] to be the only theory which candidate to be modular invariant on manifolds of the form $\tilde{T^2} \times M_4$ provided that we choose our A- and B-cycles properly. Here we should consider the case when $M_4 = S^1 \times M_3$ where S^1 is (Euclidean and periodic) time. We then combine one of the one-cycles of T^2 with the S^1-time to a new two-torus $\tilde{T^2}$. The remaining four-manifold will then contain a one-cycle. This means that the modular group of the $\tilde{T^2}$ does not constrain all the entries in α and β to be zero. But by combining modular groups from all two-tori with one cycle being the S^1-time (in the case that $M_4 = S^1 \times M_3$ with M_3 simply connected, we have the two two-tori $\tilde{T^2} = S^1 \times a$ and $\tilde{\tilde{T^2}} = S^1 \times b$) we find that all entries in α and β must be zero. Then $w^+_{a,b} \in \mathbf{Z}$ can be interpreted as winding numbers of a self-dual string that winds around the a- and b-cycles.

In our mathematical convention the four-dimensional magnetic and electric charges will, as we will see below, be given as

$$g = \sqrt{\frac{\tau_2\hbar}{2\pi}}\int_\Sigma F \quad (59)$$

$$q = \sqrt{\frac{\tau_2\hbar}{2\pi}}\int_\Sigma *F. \quad (60)$$

Under $\tau \to \tau + 1$ we want the charges of a dyon to transform as $(g, q) \to (g, q + e)$ where e is the smallest electric charge unit. In the case when $\tau_1 = 0$, we want $(g, q) \to (q, -g)$ under $\tau_2 \to \frac{1}{\tau_2}$. This explains why we have to insert a factor proportional to $\sqrt{\tau_2}$. Now we find that

$$\begin{pmatrix} g \\ q \end{pmatrix} = \sqrt{\frac{2\pi\hbar}{\tau_2}} \begin{pmatrix} \tau_2 & 0 \\ \tau_1 & 1 \end{pmatrix} \begin{pmatrix} w_a^+ \\ w_b^+ \end{pmatrix} \tag{61}$$

If we now put $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g_{YM}^2 \hbar}$, then we get

$$\begin{aligned} g &= \frac{4\pi}{g_{YM}} w_a^+ \\ q &= \frac{g_{YM}\hbar}{2} w_b^+ + \frac{\theta g_{YM}\hbar}{2\pi} w_a^+ \end{aligned} \tag{62}$$

The smallest electric charge is then $e = \sqrt{\frac{2\pi\hbar}{\tau_2}} = \frac{g_{YM}\hbar}{2}$. Now this is precisely the charge quantization one has in $N = 4$ SYM with coupling constant g_{YM} and theta angle θ and gauge group $SU(2)$ (or the dual group $SO(3)$) spontaneously broken to $U(1)$ by a Higgs vacuum expectation value [3]. That a six dimensional with one tensor multiplet coupled to massive strings reduces to an $SU(2)$ gauge theory is precisely what one should expect. More generally a theory with $N-1$ massless tensor multiplets should reduce to a theory with $N-1$ massless $U(1)$ gauge fields, arising from an $SU(N)$ gauge theory spontaneously broken by a generic Higgs field to $U(1)^{N-1}$.

I would like to thank M. Henningson for discussions.

A Appendix - Gamma matrices

A.1 The SO(1,5) spinor representaion

The chirality matrix is $\gamma = -\gamma_0\gamma_1 \cdots \gamma_5$. We let c^{AB} denote the charge conjugation matrix. It can be choosen to be either symmetric or antisymmetric. We will choose it to be symmetric. From the fact that $4 \times 4'$ contains a singlet we deduce that the invariant charge conjugation tensor must be of the form

$$c^{AB} = \begin{pmatrix} 0 & c^{\alpha\beta'} \\ c^{\alpha'\beta} & 0 \end{pmatrix} \tag{63}$$

where $c^{\alpha\beta'} = c^{\beta'\alpha}$, if we choose a representation where

$$\begin{aligned} \gamma_A{}^B &= \begin{pmatrix} \delta_\alpha^\beta & 0 \\ 0 & -\delta_{\alpha'}^{\beta'} \end{pmatrix} \\ (\gamma_\mu)_A{}^B &= \begin{pmatrix} 0 & (\gamma_\mu)_\alpha{}^{\beta'} \\ (\gamma_\mu)_\alpha{}^{\beta'} & 0 \end{pmatrix} \end{aligned} \tag{64}$$

The gamma matrices must be antisymmetric, e.g $(\gamma^\mu)_{\alpha\beta} = -(\gamma^\mu)_{\beta\alpha}$. We will raise and lower spinor indices as

$$\begin{aligned} \psi^\alpha &= c^{\alpha\beta'}\psi_{\beta'} \\ \psi^\alpha &= \psi^{\beta'}c_{\beta'\alpha} \end{aligned} \tag{65}$$

where

$$c_{\alpha\beta'}c^{\beta'\gamma} = \delta_\alpha^\gamma. \tag{66}$$

A.2 The SO(5) spinor representation

We let $\Omega^{ij} = -\Omega^{ji}$ denote the charge conjugation matrix, and we use the conventions

$$\begin{aligned} \psi^i &\equiv \Omega^{ij}\psi_j \\ \psi_i &\equiv \psi^j\Omega_{ji} \\ \psi_i\chi^i &= \psi^i\Omega_{ij}\chi^j = -\psi^i\chi_i \end{aligned} \tag{67}$$

Then Ω has to satisfy

$$\Omega^{ij}\Omega_{jk} = \Omega_{kj}\Omega^{ji} = -\delta_k^i. \tag{68}$$

We will denote the gamma matrices as $(\sigma_a)_i{}^j$.

B Appendix - Canonical quantization

We will here quantize the non-chiral theory with Lagrangian density $\mathcal{L} = -\frac{1}{2}H \wedge *H$, $H = dB$, in a curved space with Minkowski signature, and then we will also quantize the corresponding chiral theory. We will first treat B_{ij} and B_{ji} as independent fields. It is only the antisymmetric part, $\frac{1}{2}(B_{ij} - B_{ji})$, which occurs in the action. We then get the primary constraints for the canonical momentum $\Pi^{ij} = -\frac{\sqrt{|G|}}{2}H^{0ij}$

associated with B_{ij}, $\Pi^{ij}+\Pi^{ji}=0$ and $\Pi^{i0}=0$, and the secondary constraints $\partial_i\Pi^{ij}=0$. We eliminate the symmetric parts by the gauge fixing conditions $B_{ij}+B_{ji}=0$, and the $0i$-components by the gauge fixing condition $B_{i0}=0$. By imposing the gauge fixing condition $\partial^i B_{ij}=0$ we have finally fixed the gauge completely. The Poisson bracket is as always given by

$$\{B_{ij}(x),\Pi^{i'j'}(x')\}=\delta_i^{i'}\delta_j^{j'} \tag{69}$$

and for the (partially) reduced phase space variables we get the bracket (which rigorously should be computed as a Dirac-bracket. The result happens to coincides with what one gets by just antisymmetrizing the indices),

$$\{B_{[ij]}(x),\Pi^{[i'j']}(x')\}_*=\delta_{ij}^{i'j'}. \tag{70}$$

Now, after that we have reduced our phase space, we will drop the antisymmetrization symbol [ ].

The constraints we have choosen here are not independent. There are two relations between them, $\partial_i\partial_j\Pi^{ij}=0$ and $\partial^i\partial^j B_{ij}=0$. We therefore introduce two 4×5-matrices α and β of rank 4. Then the independent second class constraints can be expressed as

$$\begin{aligned}\alpha_k^I\partial_i\Pi^{ik} &= 0\\ \beta_{I'}^{k'}\partial^{i'}B_{i'k'} &= 0\end{aligned} \tag{71}$$

where $I,I'=1,2,3,4$. The matrices α and β can not be any rank 4 matrices. They are constrained by the condition that none of the above constraints are trivially fulfilled. In flat space we can work in the Fourier space. There we see that $\alpha_k^K(k_i)$ must be orthogonal to the vector space spanned by the momentum vector k_i (and similarly for β). Now the dimension of this space coincides precisely with the rank of α, so that such a matrix α (and similarly β) exists.

When we quantize we shall substitite the Dirac bracket, $\{\ ,\ \}_{**}$, on the fully reduced phase space, by the anticommutator $\frac{1}{i\hbar}[\ ,\ ]$. We thus have to compute the Dirac bracket, which is given by

$$\begin{aligned}&\{B_{ij}(x),\Pi^{i'j'}(x')\}_{**}=\{B_{ij}(x),\Pi^{i'j'}(x')\}_*\\ &-\int d^5yd^5y'\{B_{ij}(x),\alpha_l^L\partial_k\Pi^{kl}(y)\}_*C^{-1}(y,y')_L{}^{L'}\{\beta_{L'}^{l'}\partial^{k'}B_{k'l'}(y'),\Pi^{i'j'}(x')\}_*\end{aligned} \tag{72}$$

Here $C(y,y')^L{}_{L'}=\alpha_l^L\beta_{L'}^{l'}\{\partial_k\Pi^{kl}(y),\partial^{k'}B_{k'l'}(y')\}$, the exact form of which will be of no use for our purposes. By integrating by parts we get

$$\{B_{ij}(x),\Pi^{i'j'}(x')\}_{**}=\{B_{ij}(x),\Pi^{i'j'}(x')\}_*-\delta_{ij}^{kl}\delta_{k'l'}^{i'j'}\partial_k\partial^{k'}D(x,x')_l{}^{l'} \tag{73}$$

for some continuous functions $D(x,x')_l{}^{l'}$. Canonical quantization means that we should put

$$[B_{ij}(x),\Pi^{i'j'}(x')]=i\hbar\left[\delta_{ij}^{i'j'}-\delta_{ij}^{kl}\delta_{k'l'}^{i'j'}\partial_k\partial^{k'}D(x,x')_l{}^{l'}\right]. \tag{74}$$

and hence

$$\begin{aligned}[H_{ijk}(x),H^{0lm}(y)] &= -\frac{6}{\sqrt{|G(y)|}}\partial_{[i}[B_{jk]}(x),\Pi^{lm}(y)]\\ &= -i\hbar\frac{6}{\sqrt{|G(y)|}}\delta_{[jk}^{lm}\partial_{i]}\delta^5(x-y)\end{aligned} \tag{75}$$

where in the last step we have noticed that $\partial_{[i}\partial_{j]}=0$. This implies that

$$[H^+{}_{ijk}(x),H^{+0lm}(y)]=-i\hbar\frac{3}{\sqrt{|G|}}\delta_{[jk}^{lm}\partial_{i]}\delta^5(\mathbf{x}-\mathbf{y}) \tag{76}$$

or equivalently,

$$[H^+_{ijk}(x), H^+_{lmn}(y)] = i\hbar\frac{3}{2}\varepsilon_{[lmnij}\partial_{k]}\delta^5(\mathbf{x}-\mathbf{y}). \tag{77}$$

In the case that the fields are on-shell we can go one step further and rewrite this as

$$= \frac{6}{\sqrt{|G(y)|}}\partial_{[i}[B^+_{jk]}(x), \Pi^{-lm}(y)] \tag{78}$$

where we have divided the conjugate momentum into a chiral and a anti-chiral part as

$$\Pi^{\pm ij} \equiv -\frac{1}{2}\sqrt{|G|}H^{\pm 0ij} \tag{79}$$

From this we deduce that

$$[B^{\pm}_{ij}(x), \Pi^{\pm i'j'}(x')] = i\hbar\left[\frac{1}{2}\delta^{i'j'}_{ij} + \delta^{kl}_{ij}\delta^{i'j'}_{k'l'}\partial_k\partial^{k'}\tilde{D}(x,x')_l{}^{l'}\right], \tag{80}$$

where $\tilde{D}_l{}^{l'}$ are some continuous functions.

We finally show that $\exp\frac{i}{\hbar}\int\Pi^{ij}\Delta B_{ij}$ translates B to $B+\Delta B$ provided that ΔB_{ij} obey the gauge fixing constraints $\partial^k\Delta B_{kl}=0$,

$$\begin{aligned}
&\frac{i}{\hbar}[\int d^5x'\Pi^{i'j'}(x')\Delta B_{i'j'}(x'), B_{ij}(x)] = \Delta B_{ij}(x) - \int d^5x'\delta^{kl}_{ij}[\partial_k\partial^{k'}D(x,x')]\Delta B_{k'l'}(x')\\
&= \Delta B_{ij}(x) + \int d^5x\delta^{kl}_{ij}[\partial_k D(x,x')]\partial^{k'}\Delta B_{k'l'}(x')\\
&= \Delta B_{ij}(x).
\end{aligned} \tag{81}$$

C Appendix - The Dirac quantization condition and the Wilson surface

We will here obtain the Dirac quantization condition on manifolds with arbitrary topology. This we do by straightforwardly generalizing the arguments in [14]. We then let b be any three-cycle which we cover by contractible neighbourhoods U_α with no more than quadruple overlaps. We will assume the overlap regions to be contractible and let V_α be adjacent neighbourhoods obtained by contracting the overlaps. We will indicate orientation reversal with minus signs. We will define the common boundary surface $\partial V_\alpha \cap \partial V_\beta = V_\alpha \cap V_\beta$ (we could remove ∂ since these neighbourhoods were adjacent) of the boundaries ∂V_α and ∂V_β to be antisymmetric in α and β. We denote the intersection line between two such common boundaries by

$$l_{\alpha\beta\gamma} = -\partial(\partial V_\alpha \cap \partial V_\beta) \cap \partial(\partial V_\alpha \cap \partial V_\gamma) = V_\alpha \cap V_\beta \cap V_\gamma \tag{82}$$

It is totally antisymmetric. If U_α has an overlap only with U_β, U_γ and U_δ, then $\emptyset = \partial\partial V_\alpha = \partial(\partial V_\alpha \cap \partial V_\beta) + \partial(\partial V_\alpha \cap \partial V_\gamma) + \partial(\partial V_\alpha \cap \partial V_\delta)$ and we find that

$$\partial(\partial V_\alpha \cap \partial V_\beta) = l_{\alpha\beta\gamma} + l_{\alpha\beta\delta}. \tag{83}$$

Similarly we can compute that

$$\partial l_{\alpha\beta\gamma} = -\partial l_{\alpha\beta\delta} = \partial l_{\alpha\gamma\delta} = -\partial l_{\beta\gamma\delta} = V_\alpha \cap V_\beta \cap V_\gamma \cap V_\delta \tag{84}$$

which is a finite set of points.

We note that $dB_\alpha = dB_\beta$ in $U_\alpha \cap U_\beta$. We can therefore use the Poincare lemma to obtain

$$B_\alpha - B_\beta = dA_{\alpha\beta}. \tag{85}$$

in $U_\alpha \cap U_\beta$. Similarly we see that $d(A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha}) = 0$ in $U_\alpha \cap U_\beta \cap U_\gamma$ and so, by the Poincare lemma, we can write

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = df_{\alpha\beta\gamma} \tag{86}$$

in $U_\alpha \cap U_\beta \cap U_\gamma$.

Now we have all ingredients to compute the period of the field strength $H = dB$,

$$\begin{aligned}
&\int_b H = \sum_\alpha \int_{V_\alpha} dB_\alpha = \sum_\alpha \int_{\partial V_\alpha} B_\alpha \\
&= \sum_{\alpha<\beta} \int_{\partial V_\alpha \cap \partial V_\beta} (B_\alpha - B_\beta) = \sum_{\alpha<\beta} \int_{\partial V_\alpha \cap \partial V_\beta} dA_{\alpha\beta} = \sum_{\alpha<\beta} \int_{\partial(\partial V_\alpha \cap \partial V_\beta)} A_{\alpha\beta} \\
&= \sum_{\alpha<\beta<\gamma} \int_{l_{\alpha\beta\gamma}} (A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha}) = \sum_{\alpha<\beta<\gamma} \int_{l_{\alpha\beta\gamma}} df_{\alpha\beta\gamma} \\
&= \sum_{U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \cap \Sigma} (f_{\beta\gamma\delta} - f_{\alpha\gamma\delta} + f_{\alpha\beta\gamma} - f_{\alpha\beta\gamma})
\end{aligned} \tag{87}$$

Now, by definition of a connection on a gerbe, if $\frac{2\pi}{g} B$ is such a connection, then $f_{\beta\gamma\delta} - f_{\alpha\gamma\delta} + f_{\alpha\beta\gamma} - f_{\alpha\beta\gamma} \in g\mathbf{Z}$.

Now we turn to the Wilson surface. It should be something like a two-form B integrated over a two-cycle $\tilde{b}$. If $\tilde{b}$ is covered by two neighbourhoods and γ_α and γ_β are adjacent neighbourhoods then we consider the quantity

$$\int_{\gamma_\alpha} B_\alpha + \int_{\gamma_\beta} B_\beta. \tag{88}$$

This changes when we deform the neighbourhoods γ_α and γ_β such that $\delta(\gamma_\alpha + \gamma_\beta) = 0$. We get the variation

$$\int_{\delta\gamma_\alpha} (B_\alpha - B_\beta) = \int_{\delta\gamma_\alpha} dA_{\alpha\beta} = \int_{\partial\delta\gamma_\alpha} A_{\alpha\beta} \tag{89}$$

which we also can write as

$$\int_{\delta\partial\gamma_\alpha} A_{\alpha\beta} \tag{90}$$

So a sensible definition of a Wilson surface of a two-cycle which can be covered by at most two neighbourhoods is

$$\int_\gamma B = \int_{\gamma_\alpha} B_\alpha + \int_{\gamma_\beta} B_\beta - \int_{\partial\gamma_\alpha \cap \partial\gamma_\beta} A_{\alpha\beta} \tag{91}$$

In order to understand what happens for a manifold which has to be covered by three neighbourhoods we make a variation such that $\delta(\gamma_\alpha + \gamma_\beta + \gamma_\gamma) = 0$ and compute the variation

$$\begin{aligned}
&\delta\left(\int_{\gamma_\alpha} B_\alpha + \int_{\gamma_\beta} B_\beta + \int_{\gamma_\gamma} B_\gamma - \int_{\partial\gamma_\alpha \cap \partial\gamma_\beta} A_{\alpha\beta} - \int_{\partial\gamma_\alpha \cap \partial\gamma_\gamma} A_{\alpha\gamma} - \int_{\partial\gamma_\beta \cap \partial\gamma_\gamma} A_{\beta\gamma}\right) \\
&= \int_{\delta\gamma_\alpha} (B_\alpha - B_\gamma) + \int_{\delta\gamma_\beta} (B_\beta - B_\gamma) - \int_{\delta(\gamma_\alpha \cap \gamma_\beta)} A_{\alpha\beta} - \int_{\delta(\gamma_\alpha \cap \gamma_\gamma)} A_{\alpha\gamma} - \int_{\delta(\gamma_\beta \cap \gamma_\gamma)} A_{\beta\gamma} \\
&= \int_{\delta\partial\gamma_\alpha} A_{\alpha\beta} + \int_{\delta\partial\gamma_\beta} A_{\beta\gamma} - \int_{\delta(\gamma_\alpha \cap \gamma_\beta)} A_{\alpha\beta} - \int_{\delta(\gamma_\alpha \cap \gamma_\gamma)} A_{\alpha\gamma} - \int_{\delta(\gamma_\beta \cap \gamma_\gamma)} A_{\beta\gamma}
\end{aligned}$$

$$= \int_{\delta(\partial\gamma_\alpha \cap \partial\gamma_\beta)} (-A_{\alpha\beta} - A_{\beta\gamma} - A_{\gamma\alpha}) = \int_{\delta(\partial\gamma_\alpha \cap \partial\gamma_\beta \cap \partial\gamma_\gamma)} -df_{\alpha\beta\gamma}$$
$$= -\int_{\partial\delta(\partial\gamma_\alpha \cap \partial\gamma_\beta \cap \partial\gamma_\gamma)} f_{\alpha\beta\gamma} = -\delta \int_{\partial\gamma_\alpha \cap \partial\gamma_\beta \cap \partial\gamma_\gamma} f_{\alpha\beta\gamma} \tag{92}$$

We have then noticed that if $\partial(\gamma_\alpha + \gamma_\beta + \gamma_\gamma) = \emptyset$, then $\partial(\gamma_\alpha \cap \gamma_\beta) = \gamma_\alpha \cap \gamma_\beta \cap \gamma_\gamma$ since these neighbourhoods were assumed to be adjacent[2]. A sensible definition on manifolds of arbitrary topology of a Wilson surface is thus

$$\int_{\tilde{b}} B = \sum_\alpha \int_{\gamma_\alpha} B_\alpha - \sum_{\alpha<\beta} \int_{\partial\gamma_\alpha \cap \partial\gamma_\beta} A_{\alpha\beta} + \sum_{\alpha<\beta<\gamma} \int_{\partial\gamma_\alpha \cap \partial\gamma_\beta \cap \partial\gamma_\gamma} f_{\alpha\beta\gamma} \tag{93}$$

which thus is independent of how we deform the boundaries of our adjacent neighbourhoods which cover the 2-cycle $\tilde{b}$.

This definition is also nice in that it gives the same value on the Wilson surface for such two-cycles which can be covered by two neighbourhoods, as it does if we instead cover it by three neighbourhoods. But if we add a fourth neighbourhood in such a way that we get a quadruple overlap, then the Wilson surface changes. Fortunately it changes in a well-behaved way as we will see now. The simplest way to compute the change is to make use of the fact that we can continuously deform our fourth curve piece at our wish, without changing the value of the Wilson line in the way we have constructed it. We therefore choose this new curve piece γ_δ in such a way that it shrinks against the point $\gamma_\alpha \cap \gamma_\beta \cap \gamma_\delta$ so that $\gamma_\alpha \cap \gamma_\beta \cap \gamma_\delta = \gamma_\beta \cap \gamma_\gamma \cap \gamma_\delta = \gamma_\gamma \cap \gamma_\alpha \cap \gamma_\delta$. We can then express the change of the Wilson surface as

$$\int_{\alpha\cap\beta\cap\delta} f_{\alpha\beta\delta} + \int_{\alpha\cap\gamma\cap\delta} f_{\alpha\gamma\delta} + \int_{\beta\cap\gamma\cap\delta} f_{\beta\gamma\delta} - \int_{\alpha\cap\beta\cap\delta} f_{\alpha\beta\gamma}$$
$$= \int_{\alpha\cap\beta\cap\delta} (-f_{\alpha\beta\gamma} + f_{\alpha\beta\delta} - f_{\alpha\gamma\delta} + f_{\beta\gamma\delta}) \tag{94}$$

Now this quantity is an integer multiple of g. To obtain a uniquely defined Wilson surface observable we should then exponentiate,

$$\exp\left\{\frac{2\pi i}{g}\int_{\tilde{b}} B\right\}. \tag{95}$$

There is a also different way to express the Wilson surface [16]. One notices that the three-form field strength $H = dB$, when pulled back to the two-cycle $\tilde{b}$, necessarily is zero. We now cover $\tilde{b}$ with neighbourhoods U_α. Since now $dB_\alpha = 0$ in U_α, we can write $B_\alpha = d\Lambda_\alpha$. Furthermore $B_\alpha - B_\beta = dA_{\alpha\beta}$ in $U_\alpha \cap U_\beta$ so we can write $A_{\alpha\beta} - (\Lambda_\alpha - \Lambda_\beta) = df_{\alpha\beta}$. Finally we get, in $U_\alpha \cap U_\beta \cap U_\gamma$, that $df_{\alpha\beta\gamma} = A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = d(f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha})$, so $c_{\alpha\beta\gamma} \equiv f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} - f_{\alpha\beta\gamma}$ is constant. Now if we compute the Wilson surface as we have defined it in (93), and use that $B_\alpha = d\Lambda_\alpha$ in γ_α and $A_{\alpha\beta} = df_{\alpha\beta} + (\Lambda_\alpha - \Lambda_\beta)$ on $\gamma_\alpha \cap \gamma_\beta$, we find that $\int_{\tilde{b}} B = -\sum_{\alpha<\beta<\gamma} \int_{\partial\gamma_\alpha \cap \partial\gamma_\beta \cap \partial\gamma_\gamma} c_{\alpha\beta\gamma}$. Since the 'generator' $\frac{2\pi}{g} f_{\alpha\beta\gamma}$ of the U(1) gauge transformations is well-defined only modulo $2\pi\mathbf{Z}$ we immediately see that $\int_{\tilde{b}} B$ is well-defined only modulo $g\mathbf{Z}$.

[2] Of course $\partial\gamma_\alpha \cap \partial\gamma_\beta \cap \partial\gamma_\gamma$ is just a set of points. The integral then means that one should evalute the integrand in those points.

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